

TRACE FORMULAE FOR SCHRÖDINGER OPERATORS ON LATTICE

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ABSTRACT. We consider Schrödinger operators with complex decaying potentials (in general, not from trace class) on the lattice. We determine trace formulae and estimate of eigenvalues and singular measure in terms of potentials. The proof is based on estimates of free resolvent and analysis of functions from Hardy space.

1. INTRODUCTION

We consider Schrödinger operators $H = \Delta + V$ on the lattice $\mathbb{Z}^d, d \geq 3$, where Δ is the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$ given by

$$(\Delta f)(n) = \frac{1}{2} \sum_{|n-m|=1} f_m, \quad n = (n_j)_1^d \in \mathbb{Z}^d, \quad (1.1)$$

where $f = (f_n)_{n \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. We assume that the potential V satisfies

$$(Vf)(n) = V_n f_n, \quad V \in \ell^p(\mathbb{Z}^d), \quad \begin{cases} 1 \leq p < \frac{6}{5} & \text{if } d = 3 \\ 1 \leq p < \frac{4}{3} & \text{if } d \geq 4 \end{cases}. \quad (1.2)$$

Here $\ell^p(\mathbb{Z}^d), p \geq 1$ is the Banach space of sequences $f = (f_n)_{n \in \mathbb{Z}^d}$ equipped with the norm

$$\|f\|_p = \|f\|_{\ell^p(\mathbb{Z}^d)} = \begin{cases} \sup_{n \in \mathbb{Z}^d} |f_n|, & p = \infty, \\ (\sum_{n \in \mathbb{Z}^d} |f_n|^p)^{\frac{1}{p}}, & p \in [1, \infty). \end{cases}$$

It is well-known that the spectrum of the Laplacian Δ is absolutely continuous and equals

$$\sigma(\Delta) = \sigma_{\text{ac}}(\Delta) = [-d, d].$$

Note that if V satisfies (1.2), then V is the Gilbert-Schmidt operator and thus the essential spectrum of the Schrödinger operator H is given by $\sigma_{\text{ess}}(H) = [-d, d]$. The operator H has $N \leq \infty$ eigenvalues $\{\lambda_n, n = 1, \dots, N\}$ outside the interval $[-d, d]$.

We define the disc $\mathbb{D}_r \subset \mathbb{C}$ with the radius $r > 0$ by

$$\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\},$$

and abbreviate $\mathbb{D} = \mathbb{D}_1$. Define **the new spectral variable** $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2} \left(z + \frac{1}{z} \right) \in \Lambda = \mathbb{C} \setminus [-d, d], \quad z \in \mathbb{D}.$$

The function $\lambda(z)$ has the following properties (see more in Section 3):

- The function $\lambda(z)$ is a conformal mapping from \mathbb{D} onto the spectral domain Λ .
- The function $\lambda(z)$ maps the point $z = 0$ to the point $\lambda = \infty$.

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• The inverse mapping $z(\cdot) : \Lambda \rightarrow \mathbb{D}$ is given by $z = \frac{1}{d} \left(\lambda - \sqrt{\lambda^2 - d^2} \right)$, $\lambda \in \Lambda$ defined by asymptotics $z = \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3}$ as $|\lambda| \rightarrow \infty$.

Define the Hardy space $\mathcal{H}_p = \mathcal{H}_p(\mathbb{D})$, $0 < p \leq \infty$. Let F be analytic in \mathbb{D} . We say F belongs to the Hardy space \mathcal{H}_p if F satisfies $\|F\|_{\mathcal{H}_p} < \infty$, where $\|F\|_{\mathcal{H}_p}$ is given by

$$\|F\|_{\mathcal{H}_p} = \begin{cases} \sup_{0 < r < 1} \left(\frac{1}{2\pi} \int_{\mathbb{T}} |F(re^{i\vartheta})|^p d\vartheta \right)^{\frac{1}{p}} & \text{if } 0 < p < \infty, \\ \sup_{z \in \mathbb{D}} |F(z)| & \text{if } p = \infty \end{cases},$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Note that the definition of the Hardy space \mathcal{H}_p involves all $r \in (0, 1)$.

Let \mathcal{B} denote the class of bounded operators. Let \mathcal{B}_1 and \mathcal{B}_2 be the trace and the Hilbert-Schmidt classes equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$ correspondingly.

Introduce the free resolvent $R_0(k) = (\Delta - \lambda)^{-1}$, $\lambda \in \Lambda$. For $V \in \ell^2(\mathbb{Z}^d)$ we define the regularized determinant $\mathcal{D}(\lambda)$ in the cut domain Λ and the modified determinant D in the disc \mathbb{D} by

$$\begin{aligned} \mathcal{D}(\lambda) &= \det \left[(I + V R_0(\lambda)) e^{-V R_0(\lambda)} \right], \quad \lambda \in \Lambda, \\ D(z) &= \mathcal{D}(\lambda(z)), \quad z \in \mathbb{D}. \end{aligned} \tag{1.3}$$

The function \mathcal{D} is a suitable regularization of the non-defined determinant $\det(I + V R_0(\lambda))$, see [GK69]. The function \mathcal{D} is analytic in Λ and the function D is analytic in the disc \mathbb{D} . It has $N \leq \infty$ zeros (counted with multiplicity) z_1, z_2, \dots in the disc \mathbb{D} . Note that $\lambda_j = \lambda(z_j)$ is an eigenvalue of H (counted with multiplicity).

1.1. Complex potentials. In this paper we combine classical results about Hardy spaces and estimates of the free resolvent from [KM17], this gives us new trace formulae for discrete Schrödinger operators $H = \Delta + V$ on the lattice \mathbb{Z}^d , where the potential V is complex and satisfies the condition (1.2). We improve results from [KL16], where potentials are considered under the weaker condition $|V|^{\frac{2}{3}} \in \ell^1(\mathbb{Z}^d)$.

Introduce the additional conformal mapping $\varkappa : \Lambda \rightarrow \mathbb{K} = \mathbb{C} \setminus [id, -id]$ by

$$\begin{aligned} \varkappa &= \sqrt{\lambda^2 - d^2}, \quad \lambda \in \Lambda, \\ \varkappa &= \lambda - \frac{d^2}{2\lambda} + \frac{O(1)}{\lambda^3} \quad \text{as } |\lambda| \rightarrow \infty. \end{aligned} \tag{1.4}$$

Note that if $\lambda \in \Lambda$ and $\lambda \rightarrow \lambda_0 \in [-d, d]$, then we obtain $|\operatorname{Im} \lambda| + |\operatorname{Re} \varkappa(\lambda)| \rightarrow 0$.

Theorem 1.1. *Let a potential V satisfy (1.2). Then the modified determinant D is analytic in the disc \mathbb{D} and is Hölder up to the boundary and satisfies*

$$\|D\|_{\mathcal{H}_\infty(\mathbb{D})} \leq e^{C_* \|V\|_p^2/2}, \tag{1.5}$$

where the constant C_* is defined in (3.9). It has $N \leq \infty$ zeros $\{z_j\}_{j=1}^N$ in the disc \mathbb{D} , such that

$$\begin{aligned} 0 < r_0 = \inf |z_j| = |z_1| \leq |z_2| \leq \dots \leq |z_j| \leq |z_{j+1}| \leq |z_{j+2}| \leq \dots, \\ \sum_{j=1}^N \left[(1 - |z_j|) + |\operatorname{Im} \lambda_j| + |\operatorname{Re} \varkappa(\lambda_j)| \right] < \infty. \end{aligned} \tag{1.6}$$

Moreover, the function $\psi(z) = \log D(z)$ is analytic in \mathbb{D}_{r_0} and has the following Taylor series (here $a = \frac{2}{d}$)

$$\begin{aligned} \psi(z) &= -\psi_2 z^2 - \psi_3 z^3 - \psi_4 z^4 + \dots, \quad \text{as } |z| < r_0, \\ \psi_2 &= \frac{a^2}{2} \text{Tr } V^2, \quad \psi_3 = \frac{a^3}{3} \text{Tr } V^3, \quad \psi_4 = \frac{a^4}{4} \text{Tr } (V^4 + 2VH_0VH_0 + 4V^2H_0^2) - \psi_2, \dots \end{aligned} \quad (1.7)$$

For the function D we define the Blaschke product $B(z), z \in \mathbb{D}$ by: $B = 1$ if $N = 0$ and

$$B(z) = \prod_{j=1}^N \frac{|z_j|}{z_j} \frac{(z_j - z)}{(1 - \bar{z}_j z)}, \quad \text{if } N \geq 1. \quad (1.8)$$

It is well known that the Blaschke product $B(z), z \in \mathbb{D}$ given by (1.8) converges absolutely for $\{|z| < 1\}$ and satisfies $B \in \mathcal{H}_\infty$ with $\|B\|_{\mathcal{H}_\infty} \leq 1$, since $D \in \mathcal{H}_\infty$ [Koo98]. The Blaschke product B has the Taylor series at $z = 0$:

$$\begin{aligned} \log B(z) &= B_0 - B_1 z - B_2 z^2 - \dots \quad \text{as } z \rightarrow 0, \\ B_0 &= \log B(0) < 0, \quad B_1 = \sum_{j=1}^N \left(\frac{1}{z_j} - \bar{z}_j \right), \dots, \quad B_n = \frac{1}{n} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n \right), \dots \end{aligned} \quad (1.9)$$

where each B_n satisfy $|B_n| \leq \frac{2}{r_0^n} \sum_{j=1}^N (1 - |z_j|)$.

We describe the canonical representation of the determinant $D(z) = \mathcal{D}(\lambda(z)), z \in \mathbb{D}$.

Corollary 1.2. *Let a potential V satisfy (1.2). Then there exists a singular measure $\sigma \geq 0$ on $[-\pi, \pi]$, such that the determinant D has a canonical factorization for all $|z| < 1$ given by*

$$\begin{aligned} D(z) &= B(z) e^{-K_\sigma(z)} e^{K_D(z)}, \\ K_\sigma(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\sigma(t), \\ K_D(z) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |D(e^{it})| dt, \end{aligned} \quad (1.10)$$

where $\log |D(e^{it})| \in L^1(-\pi, \pi)$ and the measure σ satisfies

$$\text{supp } \sigma \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}. \quad (1.11)$$

Remarks. 1) For the canonical factorisation of analytic functions see, for example, [Koo98].

2) Note that for the inner function $D_{in}(z)$ defined by $D_{in}(z) = B(z) e^{-K_\sigma(z)}$, we have $|D_{in}(z)| \leq 1$, since $d\sigma \geq 0$ and $\text{Re} \frac{e^{it} + z}{e^{it} - z} \geq 0$ for all $(t, z) \in \mathbb{T} \times \mathbb{D}$.

3) The function $D_B = \frac{D}{B}$ has no zeros in the disk \mathbb{D} and satisfies

$$\log D_B(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad z \in \mathbb{D},$$

where the measure μ equals

$$d\mu(t) = \log |D(e^{it})| dt - d\sigma(t).$$

Theorem 1.3. (Trace formulae.) *Let V satisfy (1.2). Then the following identities hold:*

$$\frac{\sigma(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |D(e^{it})| dt \geq 0, \quad (1.12)$$

$$B_1 = \sum_{j=1}^N \left(\frac{1}{z_j} - \bar{z}_j \right) = \frac{1}{\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \quad (1.13)$$

$$\sum_{j=1}^N \left(\frac{1}{z_j^2} - \bar{z}_j^2 \right) = \frac{2}{d^2} \operatorname{Tr} V^2 + \frac{1}{\pi} \int_{\mathbb{T}} e^{-i2t} d\mu(t), \quad (1.14)$$

$$B_n = \psi_n + \frac{1}{\pi} \int_{\mathbb{T}} e^{-int} d\mu(t), \quad n = 2, 3, \dots \quad (1.15)$$

where $B_0 = \log B(0) = \log \left(\prod_{j=1}^N |z_j| \right) < 0$ and B_n are given by (1.9). In particular,

$$\sum_{j=1}^N \left(\operatorname{Re} \kappa_j + i \operatorname{Im} \lambda_j \right) = \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \quad (1.16)$$

$$\sum_{j=1}^N \left[(\kappa_{j1} \lambda_{j1} - \lambda_{j2} \kappa_{j2}) + i(\lambda_{j1} \lambda_{j2} + \kappa_{j1} \kappa_{j2}) \right] = \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-2it} d\mu(t), \quad (1.17)$$

where $\kappa_j = \sqrt{\lambda_j^2 - d^2} = \kappa_{j1} + i\kappa_{j2}$ and $\lambda_j = \lambda_{j1} + i\lambda_{j2}$.

We describe estimates of eigenvalues in terms of potentials.

Theorem 1.4. *Let V satisfy (1.2). Then we have the following estimates:*

$$\sum (1 - |z_j|) \leq -B_0 \leq \frac{C_*^2}{2} \|V\|_p^2 - \frac{\sigma(\mathbb{T})}{2\pi}, \quad (1.18)$$

$$\left| \sum_{j=1}^N (\operatorname{Re} \kappa_j + i \operatorname{Im} \lambda_j) \right| \leq d C_*^2 \|V\|_p^2, \quad (1.19)$$

$$\frac{1}{2\pi} \left| \int_{\mathbb{T}} e^{-int} d\mu(t) \right| \leq C_*^2 \|V\|_p^2, \quad \forall n \in \mathbb{Z}, \quad (1.20)$$

and in particular,

$$\begin{aligned} \sum_{j=1}^N \operatorname{Im} \lambda_j &\leq d C_*^2 \|V\|_p^2, & \text{if } \operatorname{Im} V &\geq 0, \\ \sum_{j=1}^N \operatorname{Re} \kappa_j &\leq d C_*^2 \|V\|_p^2, & \text{if } \operatorname{Re} V &\geq 0. \end{aligned} \quad (1.21)$$

1.2. Real potentials. Note that some of the results stated in above theorems are new even for real-valued potentials. We consider Schrödinger operators $H = \Delta + V$ with real potentials V under the condition (1.2). In this case all eigenvalues λ_j and the numbers $z_j, \varkappa(\lambda_j)$ for all $j = 1, \dots, N$ are real. Thus we have the same modified determinant $D(z)$ and Theorems 1.1-1.3 hold true. Then from these results we obtain trace formulae for real potentials.

Corollary 1.5. (Trace formulae.) *Let a potential V be real and satisfy (1.2). Then*

$$\frac{\sigma(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |D(e^{it})| dt \geq 0, \quad (1.22)$$

$$\sum_{j=1}^N \varkappa_j = \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \quad (1.23)$$

$$\sum_{j=1}^N \varkappa_j \lambda_j = \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-2it} d\mu(t), \quad (1.24)$$

where $B_0 = \log B(0) = \log \left(\prod_{j=1}^N |z_j| \right) < 0$ and $\varkappa_j = |\lambda_j^2 - d^2|^{\frac{1}{2}} \operatorname{sign} \lambda_j$.

From Corollary 1.5 and Theorem 1.4 we obtain

Corollary 1.6. *Let a potential V be real and satisfy (1.2). Then*

$$\frac{\sigma(\mathbb{T})}{2\pi} - B_0 \leq \frac{C_*^2}{2} \|V\|_p^2, \quad (1.25)$$

$$\left| \sum_{j=1}^N \varkappa_j \right| \leq d C_*^2 \|V\|_p^2, \quad (1.26)$$

$$\sum_{j=1}^N \varkappa_j \lambda_j \leq \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} C_*^2 \|V\|_p^2, \quad (1.27)$$

where $B_0 = \log B(0) = \log \left(\prod_{j=1}^N |z_j| \right) < 0$ and $\varkappa_j = |\lambda_j^2 - d^2|^{\frac{1}{2}} \operatorname{sign} \lambda_j$.

There are a lot of papers about eigenvalues of Schrödinger operators in \mathbb{R}^d with complex-valued potentials decaying at infinity. Bounds on sums of powers of eigenvalues were obtained in [FLLS06, LS09, DHK09, FLS16, F3, S10] and see references therein. These bounds generalise the Lieb–Thirring bounds [LT76] to the non-selfadjoint case.

For discrete Schrödinger operators most of the results were obtained for the \mathbb{Z}^1 self-adjoint case, see, for example, [T89]. For the nonself-adjoint case we mention [BGK09] and see references therein. Schrödinger operators with decreasing potentials on the lattice $\mathbb{Z}^d, d > 1$ have been considered by Boutet de Monvel-Sahbani [BS99], Isozaki-Korotyaev [IK12], Isozaki-Morioka [IM14], Kopylova [Ko10], Korotyaev-Møller [KM17], Rosenblum-Solomjak [RS09], Shaban-Vainberg [SV01] and see references therein. Scattering on other graphs was discussed by Ando [A12], Korotyaev-Saburova [KS15] and Korotyaev-Møller-Rasmussen [KMR17], Parra-Richard [PR17].

2. PRELIMINARIES

2.1. Trace class operators. We recall some well-known facts.

- Let $A, B \in \mathcal{B}$ and $AB, BA \in \mathcal{B}_1$. Then

$$\operatorname{Tr} AB = \operatorname{Tr} BA, \quad (2.1)$$

$$\det(I + AB) = \det(I + BA). \quad (2.2)$$

- Let an operator-valued function $\Omega : \mathcal{D} \rightarrow \mathcal{B}_1$ be analytic for some domain $\mathcal{D} \subset \mathbb{C}$ and $(I + \Omega(z))^{-1} \in \mathcal{B}$ for any $z \in \mathcal{D}$. Then for the function $F(z) = \det(I + \Omega(z))$ we have

$$F'(z) = F(z) \operatorname{Tr} \Omega(z)^{-1} \Omega'(z). \quad (2.3)$$

- In the case $A \in \mathcal{B}_2$ we define the modified determinant $\det_2(I + A)$ by

$$\det_2(I + A) = \det \left((I + A)e^{-A} \right). \quad (2.4)$$

The modified determinant satisfies (see (2.2) in Chapter IV, [GK69])

$$|\det_2(I + A)| \leq e^{\frac{1}{2}\|A\|_{\mathcal{B}_2}^2}, \quad (2.5)$$

and $I + A$ is invertible if and only if $\det_2(I + A) \neq 0$.

2.2. Fredholm determinant. Consider the bounded operators $V \in \mathcal{B}_2$ and H_0 acting in the Hilbert space \mathcal{H} . Define the operator $H = H_0 + V$. Introduce the resolvents

$$R_0(\lambda) = (H_0 - \lambda)^{-1}, \quad \lambda \notin \sigma(H_0) \quad \text{and} \quad R(\lambda) = (H - \lambda)^{-1}, \quad \lambda \notin \sigma(H).$$

For $V \in \mathcal{B}_2$ we define the regularized determinant \mathcal{D} by

$$\mathcal{D}(\lambda) = \det \left[(I + V R_0(\lambda)) e^{-V R_0(\lambda)} \right], \quad \lambda \notin \sigma(H_0). \quad (2.6)$$

Note that for any bounded operator H and for large λ we have

$$R(\lambda) = -\frac{1}{\lambda} \sum_{n \geq 0} \left(\frac{H}{\lambda} \right)^n, \quad |\lambda| > \|H\|, \quad (2.7)$$

where the series is absolutely convergent.

Lemma 2.1. *Let operators $V \in \mathcal{B}_2$ and $H_0 \in \mathcal{B}$ and the modified determinant $\mathcal{D}(\lambda)$ be defined by (2.6). Then $\mathcal{D}(\lambda)$ is analytic in $\{\lambda \in \mathbb{C} : |\lambda| > r_0\}$ for $r_0 = \|H_0\|$. Moreover*

$$\mathcal{D}(\lambda) = 1 + O(1/\lambda^2) \quad \text{as} \quad |\lambda| \rightarrow \infty, \quad (2.8)$$

$$\log \mathcal{D}(\lambda) = - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \operatorname{Tr} (V R_0(\lambda))^n, \quad (2.9)$$

and

$$\log \mathcal{D}(\lambda) = - \sum_{n \geq 2} \frac{d_n}{\lambda^n} = -\frac{d_2}{\lambda^2} - \frac{d_3}{\lambda^3} - \frac{d_4}{\lambda^4} - \dots, \quad (2.10)$$

$$d_2 = \frac{1}{2} \operatorname{Tr} V^2, \quad d_n = \frac{1}{n} \operatorname{Tr} (H^n - H_0^n - n H_0^{n-1} V), \quad n \geq 2,$$

where the right-hand side is uniformly convergent on $\{\lambda \in \mathbb{C} : |\lambda| \geq r\}$ for $r = \|V\| + \|H_0\| + 1$. In particular,

$$d_3 = \frac{1}{3} \text{Tr} (3V^2 H_0 + V^3), \quad d_4 = \frac{1}{4} \text{Tr} (2V H_0 V H_0 + 4V^2 H_0^2 + 4V^3 H_0 + V^4). \quad (2.11)$$

ii) The function $\psi(z) = \log \mathcal{D}(\lambda(z))$ is analytic in \mathbb{D}_r for some $r > 0$ and has the following Taylor series

$$\psi(z) = -\psi_2 z^2 - \psi_3 z^3 - \psi_4 z^4 + \dots, \quad \text{as } |z| < r, \quad (2.12)$$

and

$$\psi_2 = a^2 d_2, \quad \psi_3 = a^3 d_3, \quad \psi_4 = a^4 d_4 - \psi_2, \dots, \quad (2.13)$$

here $a = \frac{2}{d}$ and the coefficients d_n are given by (2.10).

Remark. Due to (2.8) we take the branch of $\log \mathcal{D}$ so that $\log \mathcal{D}(\lambda) = o(1)$ as $|\lambda| \rightarrow \infty$.

Proof. i) We have

$$\| (V R_0(\lambda))^2 \|_{\mathcal{B}_1} \leq \frac{\|V\|_{\mathcal{B}_2}^2}{|\lambda|^2} \quad \text{for } |\lambda| > 2r. \quad (2.14)$$

The Taylor series for the entire function e^{-T} and the estimate (2.14) give at $T = V R_0(\lambda)$

$$[(I + T)e^{-T}] = (I + T)(1 - T + T^2 O(1)) = 1 - T^2 + T^2 O(1) = I + T^2 O(1).$$

Take $r_1 > 0$ large enough. Then for $|\lambda| > r$, we have by the resolvent equation

$$R(\lambda) = R_0(\lambda) + \sum_{n=1}^{\infty} (-1)^n R_0(\lambda) \left(V R_0(\lambda) \right)^n = \sum_{n=0}^{\infty} (-1)^n R_0(\lambda) \left(V R_0(\lambda) \right)^n, \quad (2.15)$$

where the right-hand side is uniformly convergent on $\{\lambda \in \mathbb{C} : |\lambda| \geq r\}$. By (2.3), (2.14) and using (2.1), we have for $|\lambda| > r$ the following

$$\begin{aligned} \mathcal{D}'(\lambda) &= -\mathcal{D}(\lambda) \text{Tr} \left(Y(\lambda) Y_0'(\lambda) \right) = -\mathcal{D}(\lambda) \text{Tr} \left(V R(\lambda) V R_0^2(\lambda) \right) \\ &= -\mathcal{D}(\lambda) \text{Tr} \left(R_0(\lambda) V R(\lambda) V R_0(\lambda) \right) = -\mathcal{D}(\lambda) \text{Tr} \left(R(\lambda) (V R_0(\lambda))^2 \right). \end{aligned} \quad (2.16)$$

Thus (2.15) gives

$$(\log \mathcal{D}(\lambda))' = -\text{Tr} \sum_{n=0}^{\infty} (-1)^n R_0(\lambda) \left(V R_0(\lambda) \right)^{n+2} = -\text{Tr} \sum_{n=2}^{\infty} (-1)^n R_0(\lambda) \left(V R_0(\lambda) \right)^n. \quad (2.17)$$

Then integrating and using

$$\frac{d}{d\lambda} \left(\text{Tr} \left(V R_0(\lambda) \right)^n \right) = n \text{Tr} R_0(\lambda) \left(V R_0(\lambda) \right)^n$$

we obtain (2.9). The identities (2.16) and $R = R_0 - R V R_0$ imply

$$(\log \mathcal{D}(\lambda))' = -\text{Tr} \left(R(\lambda) - R_0(\lambda) + R_0(\lambda) V R_0(\lambda) \right) = -\text{Tr} \left(R(\lambda) - R_0(\lambda) + V R_0^2(\lambda) \right).$$

Using the identity (2.7) we obtain

$$(\log \mathcal{D}(\lambda))' = \sum_{n=0}^{\infty} \frac{\text{Tr} (H^n - H_0^n - n H_0^{n-1} V)}{\lambda^{n+1}} = \sum_{n=2}^{\infty} \frac{n d_n}{\lambda^{n+1}}.$$

In view of (2.8), we get (2.10), (2.11).

ii) Using (2.10), (2.11) and the identity $\lambda = \frac{d}{2}(z + \frac{1}{z})$ we obtain (2.12), (2.13). ■

Remark. Consider the case when an operator $V \in \mathbb{B}_2$ and an operator $H_0 \in \mathbb{B}$ is self-adjoint. Then each zero of $\mathcal{D}(\lambda)$ outside $\sigma(H_0)$ is eigenvalue of $H = H_0 + V$ and its multiplicity is a multiplicity of this eigenvalue.

For $\lambda \notin \sigma(H_0)$, the eigenvalue problem $(H - \lambda)u = 0$ is equivalent to $(I + (H_0 - \lambda)^{-1}V)u = 0$, which has a non-trivial solution if and only if $\mathcal{D}(\lambda) = 0$.

We will determine the asymptotics of $\log B(z)$ as $z \rightarrow 0$. For a sufficiently small z and for $t = z_j \in \mathbb{D}$ for some j we have the following identity:

$$\log \frac{|t|}{t} \frac{t - z}{1 - \bar{t}z} = \log |t| + \log \left(1 - \frac{z}{t}\right) - \log(1 - \bar{t}z) = \log |t| - \sum_{n \geq 1} \left(\frac{1}{t^n} - \bar{t}^n\right) \frac{z^n}{n}.$$

Besides,

$$|1 - |t|^n| \leq n|1 - |t||,$$

$$|t^{-n} - \bar{t}^n| \leq |1 - t^n| + |1 - t^{-n}| \leq |1 - t^n| \left(1 + \frac{1}{|t|^n}\right) \leq |1 - |t|^n| \frac{2}{r_0^n} \leq |1 - |t|| \frac{2n}{r_0^n},$$

where $r_0 = \inf |z_j| > 0$. This yields

$$\begin{aligned} \log B(z) &= \sum_{j=1}^N \log \frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z} = \sum_{j=1}^N \left(\log |z_j| + \log \left(1 - (z/z_j)\right) - \log(1 - \bar{z}_j z) \right) \\ &= \sum_{j=1}^N \log |z_j| - \sum_{n=1}^{\infty} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n\right) \frac{z^n}{n} = \log B(0) - b(z), \\ b(z) &= \sum_{n=1}^{\infty} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n\right) \frac{z^n}{n} = \sum_{n=1}^{\infty} z^n B_n, \quad B_n = \frac{1}{n} \sum_{j=1}^N \left(\frac{1}{z_j^n} - \bar{z}_j^n\right), \end{aligned} \tag{2.18}$$

where the function b is analytic in the disk $\{|z| < \frac{r_0}{2}\}$ and B_n satisfy

$$|B_n| \leq \frac{1}{n} \sum_{j=1}^N \left| \frac{1}{z_j^n} - \bar{z}_j^n \right| \leq \frac{2}{r_0^n} \sum_{j=1}^N |1 - |z_j|| = \frac{2}{r_0^n} \mathcal{Z}_D,$$

where $\mathcal{Z}_D = \sum_{j=1}^{\infty} (1 - |z_j|)$. Thus

$$|b(z)| \leq \sum_{n=1}^{\infty} |B_n| |z|^n \leq 2\mathcal{Z}_D \sum_{n=1}^{\infty} \frac{|z|^n}{r_0^n} = \frac{2\mathcal{Z}_D}{1 - \frac{|z|}{r_0}}.$$

3. COMPLEX POTENTIALS

3.1. Momentum representation. Define the Fourier transformation $\Phi : \ell^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{T}^d)$ by

$$f \rightarrow \hat{f}(k) = (\Phi f)(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} f_n e^{i(n,k)}, \quad k = (k_j)_1^d \in \mathbb{T}^d, \tag{3.1}$$

where $\mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$. We need the so-called momentum representation of the operator H :

$$\widehat{H} = \Phi H \Phi^* = \widehat{\Delta} + \mathcal{V}, \quad \widehat{\Delta} = \Phi \Delta \Phi^*, \quad \mathcal{V} = \Phi V \Phi^*, \quad (3.2)$$

$$(\widehat{\Delta}f)(k) = \widehat{\Delta}(k)f(k), \quad \widehat{\Delta}(k) = \sum_1^d \cos k_j \quad (3.3)$$

$$(\mathcal{V}f)(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \widehat{V}(k - k') \widehat{f}(k') dk', \quad k = (k_j)_1^d \in \mathbb{T}^d, \quad (3.4)$$

$$\widehat{V}(k) = \frac{1}{(2\pi)^{\frac{d}{2}}} \sum_{n \in \mathbb{Z}^d} V_n e^{i(n,k)}, \quad V_n = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{T}^d} \widehat{V}(k) e^{-i(n,k)} dk. \quad (3.5)$$

3.2. Preliminaries. We define the Hardy space in the upper half-plane. Let $F(\lambda), \lambda = \mu + i\nu \in \mathbb{C}_+$ be analytic on \mathbb{C}_+ . For $0 < p \leq \infty$ we say F belongs the Hardy space $\mathcal{H}_p = \mathcal{H}_p(\mathbb{C}_+)$ if F satisfies $\|F\|_{\mathcal{H}_p} < \infty$, where $\|F\|_{\mathcal{H}_p}$ is given by

$$\|F\|_{\mathcal{H}_p} = \begin{cases} \sup_{\nu > 0} \frac{1}{2\pi} \left(\int_{\mathbb{R}} |F(\mu + i\nu)|^p d\mu \right)^{\frac{1}{p}} & \text{if } 0 < p < \infty \\ \sup_{\lambda \in \mathbb{C}_+} |F(\lambda)| & \text{if } p = \infty \end{cases}.$$

Note that the definition of the Hardy space \mathcal{H}_p involves all $\nu = \text{Im } \lambda > 0$.

Recall that we have defined the new spectral variable $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2} \left(z + \frac{1}{z} \right) \in \Lambda = \mathbb{C} \setminus [-d, d], \quad z \in \mathbb{D}.$$

The function $\lambda(z)$ has the following properties:

- The function $\lambda(z)$ is a conformal mapping from \mathbb{D} onto the spectral domain Λ .
- $\lambda(\mathbb{D}) = \Lambda = \mathbb{C} \setminus [-d, d]$ and $\lambda(\mathbb{D} \cap \mathbb{C}_{\mp}) = \mathbb{C}_{\pm}$.
- Λ is the cut domain with the cut $[-d, d]$, having the upper side $[-d, d] + i0$ and the lower side $[-d, d] - i0$. The function $\lambda(z)$ maps the boundary: the upper semi-circle onto the lower side $[-d, d] - i0$ and the lower semi-circle onto the upper side $[-d, d] + i0$.
- The function $\lambda(z)$ maps the point $z = 0$ to the point $\lambda = \infty$.
- The inverse mapping $z(\cdot) : \Lambda \rightarrow \mathbb{D}$ is given by

$$z = \frac{1}{d} \left(\lambda - \sqrt{\lambda^2 - d^2} \right), \quad \lambda \in \Lambda,$$

$$z = \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3} \quad \text{as } |\lambda| \rightarrow \infty.$$

3.3. Proof of main theorems. We consider a Schrödinger operator $H = \Delta + V$ on $\ell^2(\mathbb{Z}^d)$. We assume that the potential V is complex and satisfies the condition (1.2). We present preliminary results. Recall that we can rewrite the modified determinant $\mathcal{D}(\lambda), \lambda \in \Lambda$ in the form

$$\mathcal{D}(\lambda) = \det \left[(I + Y_0(\lambda) e^{-Y_0(\lambda)}) \right], \quad z \in \Lambda,$$

where the operator $Y_0(\lambda)$ is given by

$$Y_0(\lambda) = |V|^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}}, \quad V^{\frac{1}{2}} = |V|^{\frac{1}{2}} e^{i \arg V}, \quad \lambda \in \Lambda = \mathbb{C} \setminus \sigma(H_0). \quad (3.6)$$

Proof of Theorem 1.1. Recall that the modified determinant $D(z) = \mathcal{D}(\lambda(z))$, $z \in \mathbb{D}$. The determinant $D(z)$, $z \in \mathbb{D}$ is well defined, since $V \in \mathcal{B}_2$. It is well known that if $\lambda_0 \in \Lambda$ is an eigenvalue of H , then $z_0 = z(\lambda_0) \in \mathbb{D}$ is a zero of D with the same multiplicity. We recall needed results from [KM17]:

Let the potential V satisfy (1.2). Then the operator-valued function $Y_0 : \mathbb{C} \setminus [-d, d] \rightarrow \mathcal{B}_2$ is analytic and Hölder continuous up to the boundary. Moreover, it satisfies

$$\|Y_0(\lambda)\|_{\mathcal{B}_2} \leq C_* \|V\|_p, \quad \forall \lambda \in \Lambda, \quad (3.7)$$

$$\|Y_0(\lambda) - Y_0(\mu)\|_{\mathcal{B}_2} \leq C_\alpha |\lambda - \mu|^\alpha \|V\|_p, \quad \forall \lambda, \mu \in \overline{\mathbb{C}}_\pm, \quad (3.8)$$

where C_α is some constant and the constant C_* is defined by

$$C_* = C_{p,d} + C_d^0 \Gamma(p, d), \quad C_{1,d} = 1, \quad C_{p,d} = p^{\frac{d(p-1)}{2p}},$$

$$\Gamma(p, d) = (3 + 2\kappa)^{\frac{d(p-1)}{p}}, \quad C_d^0 = \begin{cases} 16 \\ 4 \\ \frac{14 \cdot 2^{\frac{d}{4}}}{d-4} \end{cases}, \quad \kappa = \begin{cases} \frac{6(p-1)}{6-5p} & \text{if } d = 3 \\ \left(\frac{5p-1}{4-3p}\right)^{\frac{5p-4}{4(p-1)}} & \text{if } d = 4 \\ \frac{3d(p-1)}{3d-(2d+1)p} & \text{if } d \geq 5 \end{cases} \quad (3.9)$$

Due to results (3.7)-(3.8) the operator-valued function $Y_0(\lambda) : \mathbb{C}_\pm \rightarrow \mathcal{B}_2$ is analytic in the upper half-plane \mathbb{C}_+ and is Hölder up to the boundary. Then the determinant $\mathcal{D}(\lambda)$ is analytic in the upper half-plane \mathbb{C}_\pm and Hölder up to the boundary, and satisfies

$$\|\mathcal{D}\|_{\mathcal{H}_\infty(\mathbb{C}_\pm)} \leq e^{C_*^2 \|V\|_p^2/2}, \quad (3.10)$$

where the constant C_* is defined in (3.9). The function $\mathcal{D}(\lambda)$ has asymptotics (2.8), then all zeros of $\mathcal{D}(\lambda)$ are uniformly bounded, which yields $\sum |\operatorname{Im} \lambda_j| < \infty$.

Consider the function $f(\kappa) = \mathcal{D}(\lambda(\kappa))$, $\kappa \in \mathbb{K}$, where $\lambda(\kappa) = \sqrt{\kappa^2 + d^2}$ is the conformal mapping $\mathbb{K} \rightarrow \Lambda$. The function $f \in \mathcal{H}_\infty(\mathbb{K}_\pm)$, where $\mathbb{K}_\pm = \{\pm \operatorname{Re} z > 0\}$. Repeating arguments for the function $\mathcal{D}(\lambda)$ we obtain $\sum |\operatorname{Re} \lambda_j| < \infty$.

Thus similar arguments give that the operator-valued function $Y_0(\lambda(z)) : \mathbb{D} \rightarrow \mathcal{B}_2$ is analytic in the unit disc \mathbb{D} and is Hölder up to the boundary. Then the determinant $D(z)$ is analytic in the unit disc \mathbb{D} and Hölder up to the boundary, and satisfies (1.5).

Furthermore, due to Lemma 2.1 the function $\psi(z) = \log D(z)$ defined by $\log D(0) = 0$ is analytic in the disc \mathbb{D}_{r_0} with the radius $r_0 > 0$ defined by $r_0 = \inf |z_j| > 0$ and has the Taylor series as $|z| < r_0$ given by (1.7). ■

We now consider the canonical representation (1.10) (see, [Koo98], p. 76):

Let a function $f \in \mathcal{H}_p$, $p \geq 1$ and let B be its Blaschke product. Then f has the form

$$f(z) = B(z) e^{ic - K_\sigma(z)} e^{K_f(z)},$$

$$K_\sigma(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\sigma(t),$$

$$K_f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log |f(e^{it})| dt, \quad (3.11)$$

for all $|z| < 1$, where c is real constant and $\log |f(e^{it})| \in L^1(-\pi, \pi)$ and $\sigma = \sigma_f \geq 0$ is a singular measure on $[-\pi, \pi]$ such that $\operatorname{supp} \sigma \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}$.

We define the inner function and the outer function (after Beurling) in the disc by

$$\begin{aligned} f_{in}(z) &= B(z)e^{ic-K_\sigma(z)} && \text{the inner factor of } f, \\ f_{out}(z) &= e^{K_f(z)} && \text{the outer factor of } f, \end{aligned}$$

for $|z| < 1$. Note that we have $|f_{in}(z)| \leq 1$, since $d\sigma \geq 0$.

We describe the canonical representation of the determinant $D(z) = \mathcal{D}(\lambda(z))$, $z \in \mathbb{D}$.

Proof of Corollary 1.2. Theorem 1.1 implies $D \in \mathcal{H}_\infty$. Therefore the canonical representation (3.11) gives

$$D(z) = B(z)e^{ic-K_\sigma(z)}e^{K_D(z)}, \quad z \in \mathbb{D}. \quad (3.12)$$

In order to prove (1.10) we need to show $e^{ic} = 1$. From (3.12) at $z = 0$ we obtain

$$1 = D(0) = B(0)e^{ic-K_\sigma(0)}e^{K_D(0)}.$$

Since $B(0), K_\sigma(0), K_f(0)$ and c are real we obtain $e^{ic} = 1$. ■

We describe trace formulae.

Proof of Theorem 1.3. (The trace formulae.) Due to the canonical representation (1.10), the function $D_B(z) = \frac{D(z)}{B(z)}$ has no zeros in the disc \mathbb{D} and satisfies

$$\log D_B(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad z \in \mathbb{D}, \quad (3.13)$$

where the measure $d\mu = \log |f(e^{it})| dt - d\sigma(t)$. In order to show (1.12)–(1.15) we need the asymptotics of the Schwatz integral $\log D_B(z)$ as $z \rightarrow 0$. The following identity holds true

$$\frac{e^{it} + z}{e^{it} - z} = 1 + \frac{2ze^{-it}}{1 - ze^{-it}} = 1 + 2 \sum_{n \geq 1} (ze^{-it})^n = 1 + 2(ze^{-it}) + 2(ze^{-it})^2 + \dots \quad (3.14)$$

for all $(t, z) \in \partial\mathbb{D} \times \mathbb{D}$. Thus (3.13), (3.14) yield the Taylor series at $z = 0$:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) = \frac{\mu(\mathbb{T})}{2\pi} + \mu_1 z + \mu_2 z^2 + \mu_3 z^3 + \mu_4 z^4 + \dots \quad \text{as } |z| < 1, \quad (3.15)$$

where

$$\mu(\mathbb{T}) = \int_0^{2\pi} d\mu(t), \quad \mu_n = \frac{1}{\pi} \int_0^{2\pi} e^{-in\vartheta} d\mu(t), \quad n \in \mathbb{Z}.$$

We have the identity $\log D(z) = \log B(z) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t)$ for all $z \in \mathbb{D}_{r_0}$. Combining asymptotics (1.7), (2.18) and (3.15) we obtain (1.12)–(1.15). In particular, we have (1.14) and $-\log B(0) = \frac{\mu(\mathbb{T})}{2\pi} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{it})| dt - \frac{\sigma(\mathbb{T})}{2\pi} \geq 0$.

Recall that $\varkappa = \sqrt{\lambda^2 - d^2}$. We have the following identities for $z \in \mathbb{D}$ and $\lambda \in \Lambda$:

$$2\lambda = d\left(z + \frac{1}{z}\right), \quad dz = \lambda - \varkappa, \quad d\left(z - \frac{1}{z}\right) = -2\varkappa. \quad (3.16)$$

These identities yield

$$\begin{aligned} d\left(\frac{1}{z} - \bar{z}\right) &= 2\lambda - 2d \operatorname{Re} z, & \frac{d}{2} \operatorname{Im}\left(\frac{1}{z} - \bar{z}\right) &= \operatorname{Im} \lambda, \\ d\left(\frac{1}{z} - \bar{z}\right) &= 2\kappa + 2id \operatorname{Im} z, & \frac{d}{2} \operatorname{Re}\left(\frac{1}{z} - \bar{z}\right) &= \operatorname{Re} \kappa, \\ & & \frac{d}{2}\left(\frac{1}{z} - \bar{z}\right) &= \operatorname{Re} \kappa + i \operatorname{Im} \lambda. \end{aligned} \quad (3.17)$$

Let $\kappa_j = \sqrt{\lambda_j^2 - d^2}$. Then from (1.13) we get

$$\begin{aligned} B_1 &= \sum_{j=1}^N \left(\frac{1}{z_j} - \bar{z}_j\right) = \frac{1}{\pi} \int_{\mathbb{T}} e^{-it} d\mu(t), \\ \frac{d}{2\pi} \int_{\mathbb{T}} e^{-it} d\mu(t) &= \sum_{j=1}^N \frac{d}{2} \left(\frac{1}{z_j} - \bar{z}_j\right) = \sum_{j=1}^N \left(\operatorname{Re} \kappa_j + i \operatorname{Im} \lambda_j\right) \end{aligned}$$

and thus

$$\sum_{j=1}^N \operatorname{Re} \sqrt{\lambda_j^2 - d^2} = \frac{d}{2\pi} \int_{\mathbb{T}} \cos t d\mu(t), \quad \sum_{j=1}^N \operatorname{Im} \lambda_j = -\frac{d}{2\pi} \int_{\mathbb{T}} \sin t d\mu(t),$$

We show (1.17). The function $\lambda = \lambda(z) = \frac{d}{2}\left(z + \frac{1}{z}\right)$ satisfies

$$\begin{aligned} d\left(\frac{1}{z} + \bar{z}\right) &= d\left(\frac{1}{z} + z + (\bar{z} - z)\right) = 2\lambda - 2d \operatorname{Im} z, & \frac{d}{2} \operatorname{Re}\left(\frac{1}{z} + \bar{z}\right) &= \operatorname{Re} \lambda, \\ d\left(\frac{1}{z} + \bar{z}rt\right) &= d\left(z + 2\kappa + \bar{z}\right) = 2\kappa + 2d \operatorname{Re} z, & \frac{d}{2} \operatorname{Im}\left(\frac{1}{z} + \bar{z}\right) &= \operatorname{Im} \kappa, \\ & & \frac{d}{2}\left(\frac{1}{z} + \bar{z}\right) &= \operatorname{Re} \lambda + i \operatorname{Im} \kappa. \end{aligned} \quad (3.18)$$

Let $\kappa = \kappa_1 + i\kappa_2$ and $\lambda = \lambda_1 + i\lambda_2$. Then we obtain

$$\begin{aligned} \frac{d^2}{4} \left(\frac{1}{z^2} - \bar{z}^2\right) &= \frac{d^2}{4} \left(\frac{1}{z} - \bar{z}\right) \left(\frac{1}{z} + \bar{z}\right) \\ &= (\kappa_1 + i\lambda_2)(\lambda_1 + i\kappa_2) = (\kappa_1\lambda_1 - \lambda_2\kappa_2) + i(\lambda_1\lambda_2 + \kappa_1\kappa_2). \end{aligned} \quad (3.19)$$

Note that $\operatorname{sign} \kappa_1\lambda_1 = \operatorname{sign} \lambda_2\kappa_2$ and $\operatorname{sign} \lambda_1\lambda_2 = \operatorname{sign} \kappa_1\kappa_2$. Thus we get

$$\begin{aligned} \frac{1}{2} \operatorname{Tr} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} e^{-i2t} d\mu(t) &= \frac{d^2}{4} \sum_{j=1}^N \left(\frac{1}{z_j^2} - \bar{z}_j^2\right) \\ &= \sum_{j=1}^N \left((\kappa_{j1}\lambda_{j1} - \lambda_{j2}\kappa_{j2}) + i(\lambda_{j1}\lambda_{j2} + \kappa_{j1}\kappa_{j2})\right), \end{aligned} \quad (3.20)$$

and then

$$\begin{aligned}\sum_{j=1}^N (\varkappa_{j1} \lambda_{j1} - \lambda_{j2} \varkappa_{j2}) &= \frac{1}{2} \operatorname{Tr} \operatorname{Re} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} \cos 2td\mu(t), \\ \sum_{j=1}^N (\lambda_{j1} \lambda_{j2} + \varkappa_{j1} \varkappa_{j2}) &= \frac{1}{2} \operatorname{Tr} \operatorname{Im} V^2 + \frac{d^2}{4\pi} \int_{\mathbb{T}} \sin 2td\mu(t).\end{aligned}\tag{3.21}$$

■

Proof of Theorem 1.4. Estimates. The simple inequality $1 - x \leq -\log x$ for $\forall x \in (0, 1]$, implies $-B_0 = -B(0) = -\sum \log |z_j| \geq \sum (1 - |z_j|)$. Then substituting the last estimate and the estimate (1.5) into the first trace formula (1.12) we obtain (1.18).

We show (1.20). Let for shortness $C = C_*^2 \|V\|_p^2$. From (1.5) and (1.18) we obtain

$$\frac{1}{2\pi} \left| \int_{\mathbb{T}} e^{-int} d\mu(t) \right| \leq \frac{1}{2\pi} \int_{\mathbb{T}} \left(\frac{C}{2} dt + d\sigma \right) = \frac{C}{2} + \frac{\sigma(\mathbb{T})}{2\pi} \leq C, \quad \forall n \in \mathbb{Z}.\tag{3.22}$$

In order to determine the next two estimates we use the trace formula (1.16). From (1.16) and (3.22) we obtain

$$\left| \sum_{j=1}^N (\operatorname{Re} \varkappa_j + i \operatorname{Im} \lambda_j) \right| \leq \frac{d}{2\pi} \left| \int_{\mathbb{T}} e^{-int} d\mu(t) \right| = dC_*^2 \|V\|_p^2.\tag{3.23}$$

If $\operatorname{Im} V \geq 0$ (or $\operatorname{Re} V \geq 0$), then $\operatorname{Im} \lambda_j \geq 0$ ($\operatorname{Re} \lambda_j \geq 0$) and the estimates (3.23) gives (1.21).

■

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